ISRAEL JOURNAL OF MATHEMATICS **166** (2008), 235–238
DOI: 10.1007/s11856-008-1029-7

VIRTUAL BETTI NUMBERS OF COMPACT LOCALLY SYMMETRIC SPACES

BY

T. N. Venkataramana

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay - 400 005, INDIA.
e-mail: venky@math.tifr.res.in

ABSTRACT

We show that the virtual Betti number of a compact locally symmetric space with arithmetic fundamental group is either the Betti number of the compact dual or else is infinite.

1. Introduction

Let G be a connected non-compact linear Lie group with finite centre, such that G is simple modulo its centre. Let Γ be a torsion free cocompact arithmetic (not necessarily congruence) subgroup in G. Recall that Γ is said to be an arithmetic subgroup of G, if there is a semi-simple (simply connected) algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$ and a smooth surjective homomorphism $\pi : \mathbf{G}(\mathbb{R}) \to G$ with compact kernel such that $\pi(\mathbf{G}(\mathbb{Z}))$ is commensurable to Γ.

Let $i \geq 0$ be an integer. Consider the direct limit cohomology group

$$\mathcal{H}^i = \lim H^i(\Delta, \mathbb{C})$$

where the direct limit is over all finite index subgroups Δ in Γ; we emphasize that Γ is only assumed to be an arithmetic subgroup of G and is not assumed to be a congruence subgroup of G. The dimension of the direct limit $\mathcal{H}^i$ as a $\mathbb{C}$-vector space is called the **virtual** i-th Betti number of Γ.

Received November 25, 2006

THEOREM 1: *If the direct limit $\mathcal{H}^i$ is finite dimensional, then $\mathcal{H}^i = H^i(G_u/K, \mathbb{C})$ where G_u/K is the compact dual of the symmetric space G/K of G.*

As a special case we recover the following result of Cooper, Long and Reid (see [CLR]).

COROLLARY 1: *If M is a compact arithmetic hyperbolic 3-manifold with non-vanishing first Betti number, then M has infinite virtual first Betti number.*

Proof. Take $G = SL_2(\mathbb{C})$ in Theorem 1, and observe that the compact dual $G_u/K = S^3$ has vanishing first cohomology. ∎

The present note was motivated by the recent preprint [CLR] of Cooper, Long and Reid, where they prove Corollary 1 in a nice geometric context, by using crucially the fact that M is a hyperbolic 3-manifold. We show in Theorem 1 that the infiniteness of virtual Betti number is true in greater generality. The point of Theorem 1 is that the group Γ is not assumed to be a congruence subgroup; if Γ is a congruence subgroup, this is a result of A. Borel (see [B]).

2. Proof of Theorem 1

Let $K \subset G$ be a maximal compact subgroup; write $\mathfrak{k}$ and $\mathfrak{g}$ for the complexified Lie algebras of K and G. We have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Note that Γ (and hence the finite index subgroup Δ) is torsion-free and cocompact in G. We then get by the Matsushima–Kuga formula (see Chap. VII, Corollary (3.3) and Remark (3.5) (i) of [BoW] and also Chap. II, (3.4) of [BoW]),

$$H^i(\Delta, \mathbb{C}) = \mathrm{Hom}_K\left(\bigwedge^i \mathfrak{p}, \mathcal{C}^\infty(\Delta\backslash G)(0)\right).$$

In this formula, $\mathcal{C}^\infty(\Delta\backslash G)(0)$ denotes the space of complex valued smooth functions on the manifold $\Delta\backslash G$ which are annihilated by the Casimir of $\mathfrak{g}$ (the latter space in the Matsushima–Kuga formula may be identified with the space of **harmonic** differential forms of degree i on $\Delta\backslash G/K$ with respect to the G-invariant metric on the symmetric space G/K).

Taking direct limits in the Matsushima–Kuga formula yields the equality

$$\mathcal{H}^i = \lim H^i(\Delta, \mathbb{C}) = \mathrm{Hom}_K\left(\bigwedge^i \mathfrak{p}, \bigcup_{\Delta\subset\Gamma} \mathcal{C}^\infty(\Delta\backslash G)(0)\right).$$

Here, Δ runs through finite index subgroups of Γ. Consider the space

$$\mathcal{F} = \bigcup_{\Delta \subset \Gamma} \mathbb{C}^{\infty}(\Delta \backslash G)(0).$$

On the space $\mathcal{F}$ (the space of smooth functions on G annihilated by the Casimir and invariant under some finite index subgroup Δ for varying Δ), G acts on the right (since the Casimir commutes with the G-action).

Now, Γ is an arithmetic subgroup of G. As in the introduction, there exists an algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$ with a smooth surjective homomorphism $\pi : \mathbf{G}(\mathbb{R}) \to G$ such that the image $\pi(\mathbf{G}(\mathbb{Z})$ is commensurate to Γ. We define $G(\mathbb{Q})$ simply to mean the image group $\pi(\mathbf{G}(\mathbb{Q}))$. It follows from weak approximation ([PR]) that $G(\mathbb{Q})$ is dense in G.

Now, there is an action on $\mathcal{F}$ by $G(\mathbb{Q})$ on the left (which, therefore, commutes with the right G action), as follows. Given a function $\phi \in \mathcal{F}$ and given an element $g \in G(\mathbb{Q})$, the function ϕ is left Δ-invariant for some finite index subgroup Δ in Γ. Consider the function $g(\phi) = x \mapsto \phi(g^{-1}x)$. This function is left-invariant under $g\Delta g^{-1}$ and hence under $\Gamma \cap g\Delta g^{-1}$; since $g \in G(\mathbb{Q})$, it follows that g commensurates Γ and hence that the subgroup $\Gamma \cap g\Delta g^{-1}$ is of finite index in Γ. Therefore, $g(\phi)$ lies in $\mathcal{F}$. This defines an action of $G(\mathbb{Q})$ on the direct limit $\mathcal{H}^i$. Note that under this action, the action of Δ on the cohomology group $H^i(\Delta, \mathbb{C})$ is trivial.

Suppose that $\mathcal{H}^i$ is finite dimensional. Since $\mathcal{H}^i$ is a direct limit of finite dimensional vector spaces, it follows that it coincides with one of them. Therefore, there exists a finite index subgroup Δ of Γ such that

$$\mathcal{H}^i = H^i(\Delta, \mathbb{C}).$$

The last sentence of the foregoing paragraph says that while $G(\mathbb{Q})$ acts on $H^i(\Delta, \mathbb{C})$, the action by Δ is trivial. Hence the action by the normal subgroup N generated by Δ in $G(\mathbb{Q})$ is also trivial. The density of $G(\mathbb{Q})$ in G is easily seen to imply the density of the normal subgroup N in G. Thus, the image of $\bigwedge^i \mathfrak{p}$ under any element of $\mathcal{H}^i$ (viewed via the Matsushima-Kuga formula as a (K-equivariant) homomorphism of $\bigwedge^i \mathfrak{p}$ into $\mathcal{F}$), goes into (N-invariant, and by the density of N in G, into) G invariant functions in $\mathcal{C}^{\infty}(\Delta \backslash G)$, i,e, the constant functions. But $\mathrm{Hom}_K(\bigwedge^i \mathfrak{p}, \mathbb{C})$ is the space of harmonic differential forms on the compact dual G_u/K, and is therefore isomorphic to $H^i(G_u/K, \mathbb{C})$.

This proves Theorem 1.

Remark: If Γ and all the subgroups Δ are **congruence** subgroups, then one sees at once from strong approximation, that the above $G(\mathbb{Q})$ action on the direct limit translates into the action of the "Hecke Operators" $G(\mathbb{A}_f)$ ($\mathbb{A}_f$ are the ring of finite adeles) and amounts to the proof of Borel in [B]. In this sense, the proof of Theorem 1 is an extension of Borel's proof to the non-congruence case.

ACKNOWLEDGEMENTS. The author gratefully acknowledges the hospitality of the Forschungsinstitut für Mathematik, ETH, Zurich in November 2006, where this note was written. The author also thanks the referee for helpful suggestions.

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